

# $\mathbf{Z}_p$ -invariant Nonsingular Quartic Surfaces, $p \geq 5$

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## Abstract

All  $\mathbf{Z}_p$ -invariant nonsingular quartic surfaces in  $\mathbf{P}^3(k)$  are obtained for any prime  $p \geq 5$ , up to projective equivalence. As a result, if  $G$  is the projective automorphism group of a nonsingular quartic surface in  $\mathbf{P}^3(k)$ , then  $|G| = 2^a 3^b 5^c 7^d$  with  $c, d \leq 1$ .

Keywords: Algebraic surfaces, Field of characteristic zero, Automorphism groups.

Mathematics Subject Classification (2010): 14J50

## 0 Introduction

Let  $k$  be an algebraically closed field of characteristic zero, and  $n \geq 2$  an integer. Defining  $f_A \in k[x_1, \dots, x_n]$  for  $A \in GL_n(k)$  and  $f \in k[x_1, \dots, x_n]$  to be  $f(\sum_{j=1}^n \alpha_{1j} x_j, \dots, \sum_{j=1}^n \alpha_{nj} x_j)$ , where  $[\alpha_{ij}] = A^{-1}$ , we obtain a group action of  $GL_n(k)$  on the polynomial ring  $k[x] : A \cdot f = f_A$ , namely  $(AB) \cdot f = A \cdot (B \cdot f)$ , or equivalently,  $(f_A)_B = f_{AB}$ . We denote the  $(n-1)$ -dimensional projective space over  $k$  by  $\mathbf{P}^{n-1}(k)$ , a point of  $\mathbf{P}^{n-1}(k)$  whose homogeneous coordinates are  $[x_1, \dots, x_n]$  by  $(x_1, \dots, x_n)$ . Denote by  $(A)$  the projective transformation of  $\mathbf{P}^{n-1}(k)$  defined by  $A \in GL_n(k)$ , and let  $PGL_n(k)$  be the group of all projective transformations of  $\mathbf{P}^{n-1}(k)$ . Obviously  $PGL_n(k)$  acts on the power set of  $\mathbf{P}^{n-1}(k)$ . Therefore, for any subset  $S$  of  $\mathbf{P}^{n-1}(k)$  the set  $\{(A) \in PGL_n(k) : (A)S = S\}$  is a subgroup of  $PGL_n(k)$ , which we denote by  $\text{Paut}(S)$  and call the projective automorphism group of  $S$ . Let  $h \in k[x]$  be a homogeneous polynomial of degree  $d \geq 0$ . When  $G$  is a subgroup of  $PGL_n(k)$ , we say that  $h$  is  $G$ -invariant if  $h_A \sim h$ , i.e.  $h_A = \lambda h$  for some  $\lambda \in k^*$ , which may depend on  $(A)$ , for any  $(A) \in G$ . By definition  $h$  is  $\mathbf{Z}_p$ -invariant if and only if there exists a subgroup  $G$  of  $PGL_n(k)$  isomorphic to  $\mathbf{Z}_p$  such that  $h$  is  $G$ -invariant. The projective automorphism group  $\text{Paut}(h)$  of  $h$ , is the set of all  $(A)$  such that  $h_A \sim h$ . Clearly  $\text{Paut}(h) = \text{Paut}(V(h))$  if  $h$  is irreducible, in particular if the hypersurface  $V(h)$  in  $\mathbf{P}^{n-1}(k)$  is nonsingular. We shall show that any  $\mathbf{Z}_5$ -invariant nonsingular quartic form in  $k[x, y, z, t]$  takes the form, up to projective equivalence,

$$x^3y + y^3z + z^3t + t^3x + \mu x^2z^2 + \nu y^2t^2 + \lambda xyzt,$$

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\*The research was supported by the Italian Ministero dell'Istruzione, dell'Università e della Ricerca (MIUR) and by the Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni (GNSAGA).

where  $[\mu, \nu, \lambda] \in k^3$  is not the zero of the following polynomial  $R \in k[u, v, w]$ ;

$$\{(27u^2 + 6uv + 27v^2 - w^2) - 36(u + v)uvw + (u + v)w^3 + (-16u^3v^2 + 8u^2v^2w^2 - uvw^4)\}^2 - \{16 - 18(u + v)w + 48u^2v^2 + 20uvw^2\}^2$$

Note that  $\mathbf{A}_5$ -invariant nonsingular quartic forms [2] are  $\mathbf{Z}_5$ -invariant. We shall show that any  $\mathbf{Z}_7$ -invariant nonsingular quartic form in  $k[x, y, z, t]$  takes the form, up to projective equivalence,

$$x^3y + y^3z + z^3x + t^4 + \lambda xyz t,$$

where  $\lambda^4 \neq 4^4$ . We shall also show that there exists no  $\mathbf{Z}_p$ -invariant nonsingular quartic form in  $k[x, y, z, t]$  for any prime  $p > 7$ .

As a by-product the order of any projective automorphism group  $G$  of the nonsingular quartic surface is of the form  $2^a 3^b 5^c 7^d$ , where  $c, d \leq 1$ .

Finally we shall determine the projective automorphism group of the  $\mathbf{Z}_5$ -invariant quartic surface  $V(x^3y + y^3z + z^3t + t^3x)$ .

## 1 Preliminaries

Let  $d, n-1, r \geq 1$  be integers and  $\delta \in k^*$  with  $\text{ord}(\delta) = r$ . If  $A \in GL_n(k)$  is diagonal and of order  $r$ , it takes the form  $\text{diag}[\delta^{i_1}, \dots, \delta^{i_n}]$  for some integers  $i_1, \dots, i_n \in [0, r-1]$ , so that  $M_{A^{-1}} = \delta^j M$  for any monomial  $M \in k[x]$  with certain  $j \in [0, r-1]$ . Therefore the set  $\mathcal{M}_d$  of all monomials of degree  $d$  is the disjoint union of  $\mathcal{M}_d(j) = \{M \in \mathcal{M}_d : M_{A^{-1}} = \delta^j M\}$  ( $j \in [0, r-1]$ ). Hence we have

**Lemma 1.1.** *Let  $A$  and  $\mathcal{M}_d(j)$  be as above. Then a non-zero homogeneous polynomial  $f \in k[x]$  of degree  $d$  satisfies  $f_{A^{-1}} \sim f$ , if and only if  $f$  is a non-zero linear combination of monomials in  $\mathcal{M}_d(j)$  for some  $j$  such that  $\mathcal{M}_d(j) \neq \emptyset$ .*

Let  $A$  be as in Lemma 1.1 and  $M_{A^{-1}} = \delta^j M$ , where  $M$  is a monomial in  $k[x_1, \dots, x_n]$  and  $j \in [0, r-1]$ . We call  $j$ , which is considered to belong to  $\mathbf{Z}/r\mathbf{Z}$ , the index of  $M$  for  $A$ . Monomials  $x_i x_j^{d-1}$  ( $i, j \in [1, n]$ ) are called singularity-checking monomials of  $d$ -forms in  $k[x]$ . If  $d \geq 3$ , there exist  $n^2$  such monomials. Obviously any  $(d-1)$ -form not containing  $x_j^{d-1}$  vanishes at  $\varepsilon_j$ , where  $\varepsilon_j$  stands for the  $j$ -th row of the unit matrix  $E_n \in GL_n(k)$ . Let  $d \geq 2$ , and  $i, j \in [1, n]$ . A monomial  $M \in k[x]$  satisfies  $M_{x_i} \sim x_j^{d-1}$  and  $M_{x_i} \neq 0$  if and only if  $M = x_i x_j^{d-1}$ . Therefore the following lemma holds.

**Lemma 1.2.** *Let  $d \geq 2$ , and assume that a non-zero  $d$ -form  $f \in k[x]$  contains none of  $n$  forms  $x_i x_j^{d-1}$  ( $i \in [1, n]$ ) for some  $j \in [1, n]$ , then  $f_{x_i}(\varepsilon_j) = 0$  for any  $i$ , namely the projective algebraic set  $V(f)$  is singular at  $(\varepsilon_j)$ .*

**Corollary 1.3.** *Let  $d \geq 3$  and  $f \in k[x_1, \dots, x_n]$  a non-zero  $d$ -form. If  $f$  contains at most  $n-1$  singularity-checking monomials of degree  $d$ , then the projective algebraic set  $V(f)$  is singular at  $(\varepsilon_\ell) \in \mathbf{P}^{n-1}(k)$  for some  $\ell \in [1, n]$ .*

*Proof.* Let  $X = [x_{ij}] \in M_{n,n}(k[x])$  such that  $x_{ij} = x_i x_j^{d-1}$  ( $i, j \in [1, n]$ ). Then the matrix components of  $X$  are distinct. By the assumption  $f$  contains no elements of a certain column of  $X$ , the  $\ell$ -th column say. Thus  $V(f)$  is singular at  $(\varepsilon_\ell)$ .

From now on until the end of this section let  $\varepsilon \in k^*$  be of order  $p$ , a prime number not less than 5. We consider a subgroup  $G$  of  $PGL_4(k)$  isomorphic to  $\mathbf{Z}_p$ , i.e. the additive group  $\mathbf{Z}/p\mathbf{Z}$ .  $G$  has a generator  $(A)$ , where  $A \in GL_4(k)$  is of order  $p$ . We may assume  $A = \text{diag}[\varepsilon^i, \varepsilon^j, \varepsilon^\ell, \varepsilon^m]$  with  $\text{gcd}(i, j, \ell, m) = 1$  ( $i, j, \ell, m \in [0, p-1]$ ). Since

$(A) = (\text{diag}[1, \varepsilon^{(j-i)}, \varepsilon^{(\ell-i)}, \varepsilon^{(m-i)}])$  in  $PGL_4(k)$ , we may assume  $A = \text{diag}[1, \varepsilon^i, \varepsilon^j, \varepsilon^\ell]$  with  $0 \leq i \leq j \leq \ell < p$  and  $\gcd(i, j, \ell) = 1$ . If  $i = j = 0$ , we may assume  $A = D_0$ , where  $D_0 = \text{diag}[1, 1, \varepsilon, 1]$ , for  $(\varepsilon^\ell)^m = \varepsilon$  for some  $m \in [1, p-1]$ . If  $i = 0 < j = \ell$ , then we may assume  $A = D_1$ , where  $D_1 = \text{diag}[1, 1, \varepsilon, \varepsilon]$ . If  $i = 0 < j < \ell$ , we may assume  $A = D_\ell$ , where  $D_\ell = \text{diag}[1, 1, \varepsilon, \varepsilon^\ell]$  with  $1 < \ell < p$ , for  $(\varepsilon^j)^m = \varepsilon$  for some  $m \in [1, p-1]$ . Suppose  $i > 0$ . Let  $I = \{i, j, \ell\}$ . If  $|I| = 1$ , then  $G$  is conjugate to  $\langle(D_0)\rangle$ , so we may assume  $A = D_0$ . If  $|I| = 2$ , we may assume  $A = D_\ell$ . Finally suppose  $|I| = 3$ . Then we may assume  $A = D_{j,\ell}$ , where  $D_{j,\ell} = \text{diag}[1, \varepsilon, \varepsilon^j, \varepsilon^\ell]$  ( $1 < j < \ell < p$ ). Thus we have shown

**Lemma 1.4.** *Let  $G$  be a subgroup of  $PGL_4(k)$  isomorphic to  $\mathbf{Z}_p$ , and let  $D_0, D_1, D_\ell$  ( $1 < \ell < p$ ), and  $D_{j,\ell}$  ( $1 < j < \ell < p$ ) be as above. Then  $G$  is conjugate to one of the cyclic groups  $\langle(D_0)\rangle, \langle(D_1)\rangle, \langle(D_\ell)\rangle$ , and  $\langle(D_{j,\ell})\rangle$ .*

Assume  $p = 5$ . Since  $D_2^2 = \text{diag}[1, 1, \varepsilon^2, \varepsilon]$ ,  $\langle(D_2)\rangle$  and  $\langle(D_3)\rangle$  are conjugate in  $PGL_4(k)$ , namely  $\langle(D_2)\rangle \cong \langle(D_3)\rangle$ . The equalities

$$D_{3,4}^2 = \text{diag}[1, \varepsilon^2, \varepsilon, \varepsilon^3], \quad \text{and} \quad D_{2,4} = \varepsilon^4 \text{diag}[\varepsilon, \varepsilon^2, \varepsilon^3, 1]$$

imply  $\langle(D_{3,4})\rangle \cong \langle(D_{2,3})\rangle$  and  $\langle(D_{2,4})\rangle \cong \langle(D_{2,3})\rangle$ , respectively. Therefore we have

**Lemma 1.5.** *A subgroup of  $PGL_4(k)$  isomorphic to  $\mathbf{Z}_5$  is conjugate to one of the five cyclic subgroups  $\langle(D_0)\rangle, \langle(D_1)\rangle, \langle(D_2)\rangle, \langle(D_4)\rangle$  and  $\langle(D_{2,3})\rangle$ .*

Finally assume  $p = 7$ . The equalities  $D_2^4 = \text{diag}[1, 1, \varepsilon^4, \varepsilon]$  and  $D_3^5 = \text{diag}[1, 1, \varepsilon^5, \varepsilon]$  imply  $\langle(D_4)\rangle \cong \langle(D_2)\rangle$  and  $\langle(D_5)\rangle \cong \langle(D_3)\rangle$ , respectively. Since  $D_{2,4}^5 = \varepsilon^5 \text{diag}[\varepsilon^2, 1, \varepsilon^5, \varepsilon]$  and  $D_{2,6} = \varepsilon^6 \text{diag}[\varepsilon, \varepsilon^2, \varepsilon^3, 1]$ , we have  $\langle(D_{2,5})\rangle \cong \langle(D_{2,4})\rangle$  and  $\langle(D_{2,6})\rangle \cong \langle(D_{2,3})\rangle$ . Moreover, the equalities

$$\begin{aligned} D_{3,4}^2 &= \text{diag}[1, \varepsilon^2, \varepsilon^6, \varepsilon], \quad D_{3,5}^3 = \text{diag}[1, \varepsilon^3, \varepsilon^2, \varepsilon], \quad D_{3,6}^5 = \text{diag}[1, \varepsilon^5, \varepsilon, \varepsilon^2], \\ D_{4,5}^2 &= \text{diag}[1, \varepsilon^2, \varepsilon, \varepsilon^3], \quad D_{4,6}^2 = \text{diag}[1, \varepsilon^2, \varepsilon, \varepsilon^5] \end{aligned}$$

imply, respectively,

$$\begin{aligned} \langle(D_{3,4})\rangle &\cong \langle(D_{2,3})\rangle, \quad \langle(D_{3,5})\rangle \cong \langle(D_{2,3})\rangle, \quad \langle(D_{3,6})\rangle \cong \langle(D_{2,5})\rangle, \\ \langle(D_{4,5})\rangle &\cong \langle(D_{2,3})\rangle, \quad \langle(D_{4,6})\rangle \cong \langle(D_{2,5})\rangle. \end{aligned}$$

Since  $D_{5,6}^3 = \text{diag}[1, \varepsilon^3, \varepsilon, \varepsilon^4]$ , we also have  $\langle(D_{5,6})\rangle \cong \langle(D_{3,4})\rangle \cong \langle(D_{2,6})\rangle$ . Thus we arrive at

**Lemma 1.6.** *A subgroup of  $PGL_4(k)$  isomorphic to  $\mathbf{Z}_7$  is conjugate to one of the seven cyclic groups  $\langle(D_0)\rangle, \langle(D_1)\rangle, \langle(D_2)\rangle, \langle(D_3)\rangle, \langle(D_6)\rangle, \langle(D_{2,3})\rangle$  and  $\langle(D_{2,4})\rangle$ .*

## 2 $\mathbf{Z}_5$ -invariant nonsingular quartic surfaces

We shall describe  $\mathbf{Z}_5$ -invariant nonsingular quartic forms in  $k[x, y, z, t]$ . Let  $\varepsilon \in k^*$  be of order 5, the diagonal matrices  $D_j$  and  $D_{j,\ell}$  be as in the previous section, and  $A_0 = D_0, A_1 = D_1, A_2 = D_2, A_3 = D_4$  and  $A_4 = D_{2,3}$ . Denote by  $f^{[i,j]}(x, y, z, t)$  a quartic form such that  $f_{A_i^{-1}}^{[i,j]} = \varepsilon^i f^{[i,j]}$  ( $i, j \in [0, 4]$ ). Computing indices of the singularity-checking quartic monomials for  $A_i$ , we can easily show that  $f^{[i,j]}$  is singular, i.e. the projective algebraic set  $V(f^{[i,j]})$  has a singular point, except for  $[i, j] = [5, 1]$ . Meanwhile  $f^{[5,1]}$  takes the form

$$ax^3y + by^3t + cz^3x + dt^3z + \mu'x^2t^2 + \nu'y^2z^2 + \lambda'xyzt,$$

where  $abcd \neq 0$ , for  $f^{[5,1]}$  is nonsingular. There exists a nonsingular diagonal matrix  $D$  such that  $f' = f_{D^{-1}}^{[5,1]}$  takes the form

$$x^3y + y^3t + z^3x + t^3z + \mu x^2z^2 + \nu y^2t^2 + \lambda xyz t.$$

Let  $T = [e_1, e_2, e_4, e_3]$ , where the column vector  $e_i \in k^4$  ( $i \in [1, 4]$ ) is the  $i$ -th column vector of the unit matrix  $E_4 \in GL_4(k)$ . Now  $f = f_{T^{-1}}'$  takes the form

$$x^3y + y^3z + z^3t + t^3x + \mu x^2z^2 + \nu y^2t^2 + \lambda xyz t.$$

We denote this quartic form by  $f^{\mu, \nu, \lambda}$ . Clearly  $f_{A^{-1}}^{\mu, \nu, \lambda} = \varepsilon f^{\mu, \nu, \lambda}$ , where  $A = T^{-1}A_4T = \text{diag}[1, \varepsilon, \varepsilon^3, \varepsilon^2]$ .

**Proposition 2.1.** *Let  $G$  be the projective automorphism group of a nonsingular quartic form. In the decomposition  $\Pi p^{\nu(p)}$  of  $|G|$  into prime factors it holds that  $\nu(5) \leq 1$ .*

*Proof.* As is known, the projective automorphism group of a nonsingular  $d$ -form is a finite group [6] if  $d \geq 3$ . Let  $G = \text{Paut}(f)$ ,  $p = 5$  and  $c = \nu(p)$ . Suppose  $c > 1$ . Then  $G$  contains a abelian subgroup  $H$  of order  $p^2$ . We may assume  $f = f^{\mu, \nu, \lambda}$  for some  $\mu, \nu, \lambda \in k$ , hence  $(A) \in H$ , where  $A = \text{diag}[1, \varepsilon, \varepsilon^3, \varepsilon^2]$ . We can directly see that any  $X \in GL_4(k)$  such that  $AX \sim XA$ , namely  $(A)(X) = (X)(A)$  in  $PGL_4(k)$ , is diagonal. Consequently, if  $(B) \in H$ , then  $B = \text{diag}[1, \beta, \gamma, \delta]$  and  $f_{B^{-1}} \sim f$ , in particular,  $\beta = \beta^3\gamma = \gamma^3\delta = \delta^3$ , hence  $B = \text{diag}[1, \beta, \beta^{-2}, \beta^7]$  with  $\beta^{20} = 1$ . Thus  $|H| \leq 20 < p^2$ , a contradiction.

We shall find a condition for  $f^{\mu, \nu, \lambda}$  to be singular. We discuss first the case  $\mu\nu\lambda = 0$ . Note that  $f^{0,0,0}$  is nonsingular.

**Lemma 2.2.** *The quartic form  $f^{\mu,0,0}$  is singular if and only if  $\mu^4 = 3^{-6}4^4$ . The quartic form  $f^{0,\nu,0}$  is singular if and only if  $\nu^4 = 3^{-6}4^4$ .*

*Proof.* We write  $f$  for  $f^{\mu,0,0}$  and assume  $\mu \neq 0$ . Assume  $V(f)$  has a singular point at  $(x, y, z, t)$ . It can be easily seen that  $xyzt \neq 0$ . We may assume  $t = 1$ . The condition  $f_x = f_y = f_z = f_t = 0$  is equivalent to  $g_j = 0$  ( $j \in [1, 4]$ ), where  $g_1 = xf_x - zf_z$ ,  $g_2 = yf_y - zf_z$ ,  $g_3 = zf_z - f_t$ ,  $g_4 = f_t$ . The condition  $g_1 = g_2 = g_4 = 0$  yields  $y^3z = x$  and  $x^3y = z^3$ . In particular  $y^{10} = 1$ . Now that  $y^{10} = 1$  and  $x = y^3z$ , the condition  $g_j = 0$  ( $j \in [1, 4]$ ) is equivalent to

$$2\mu z^3 + 3y^4z^2 + y^7 = 0, \quad z^2 + 3y^3 = 0.$$

Since  $3(2\mu z^3 + 3y^4z^2 + y^7) - (z^2 + 3y^3) = 2z^2(3\mu z + 4y^4)$ , this condition equivalent to  $z^2 + 3y^3 = 0$  and  $3\mu z + 4y^4 = 0$ . Thus  $f$  is singular if and only if  $(4y^4/3\mu)^2 + 3y^3 = 0$ , namely  $y^5(4/3\mu)^2 = -3$ , for some  $y$  satisfying  $y^{10} = 1$ . Consequently  $f$  is singular if and only if  $(4/3\mu)^4 = 3^2$ . The second assertion follows from the equality  $f_{T^{-1}}^{\mu,0,0} = f^{0,\mu,0}$ , where  $T = [e_4, e_1, e_2, e_3] \in GL_4(k)$ .

**Lemma 2.3.** *The quartic form  $f^{0,0,\lambda}$  is singular if and only if  $\lambda^4 = 4^4$ .*

*Proof.* We write  $f$  for  $f^{0,0,\lambda}$ . Assume that  $V(f)$  is singular at  $(x, y, z, t)$ . It follows easily that  $xyzt \neq 0$ . We may assume  $t = 1$ . The condition  $f_x = f_y = f_z = f_t = 0$  is equivalent to  $g_j = 0$  ( $j \in [1, 4]$ ), where

$$g_1 = xf_x - f_t, \quad g_2 = yf_y - f_t, \quad g_3 = zf_z - f_t, \quad g_4 = f_t.$$

Let  $g'_2 = g_2 - 3g_3$ . Clearly the condition  $g_j = 0$  ( $j \in [1, 4]$ ) is equivalent to  $g_1 = g'_2 = g_3 = g_4 = 0$ , and the condition  $g_1 = g'_2 = 0$  implies  $x = z^3$  and  $x^3y = x$ , hence  $y = z^{-6}$ . The condition  $g_3 = 0$  yields  $y^3z = z^3$  so that  $z^{20} = 1$ . Let  $\eta \in k^*$  with  $\text{ord}(\eta) = 20$ , and  $z = \eta^i$ ,  $x = \eta^{3i}$  and  $y = \eta^{-6i}$  ( $i \in [0, 19]$ ). Then the condition  $g_1 = g'_2 = g_3 = g_4 = 0$  is equivalent to  $\lambda + 4\eta^{5i} = 0$ . Since  $\text{ord}(\eta^5) = 4$ ,  $V(f)$  has a singular point if and only if  $\lambda^4 = 4^4$ .

**Lemma 2.4.** (1) The quartic form  $f^{\mu,0,\lambda}$  is singular if and only if  $(16 - 18\mu\lambda)^2 - (27\mu^2 - \lambda^2 + \mu\lambda^3)^2 = 0$ .  
(2) The quartic form  $f^{0,\nu,\lambda}$  is singular if and only if  $(16 - 18\nu\lambda)^2 - (27\nu^2 - \lambda^2 + \nu\lambda^3)^2 = 0$ .

*Proof.* It suffices to prove (1). If  $\mu\lambda = 0$ , we are done by Lemma 2.2 and Lemma 2.3. Assume  $\mu\lambda \neq 0$ . We write  $f$  for  $f^{\mu,0,\lambda}$ . Suppose  $V(f)$  is singular at  $(x, y, z, t)$ . We can show  $xyzt \neq 0$ , so we may assume  $t = 1$ . The condition  $f_x = f_y = f_z = f_t = 0$  is equivalent to  $g_j = 0$  ( $j \in [1, 4]$ ), where  $g_1 = xf_x - f_t, g_2 = yf_y - f_t, g_3 = zf_z - f_t, g_4 = f_t$ . Since  $g_1 = g_2 - 3g_3 = 0$ , we obtain  $\mu x^2 z^2 + z^3 - x = 0$ . Now the condition  $g_1 = g_2 = 0$  yields  $x^3 y = z^3$  and  $x = y^3 z$ , therefore  $y^{10} = 1$ . Now that  $y^{10} = 1$  and  $x = y^3 z$ , the condition  $g_j = 0$  ( $j \in [1, 4]$ ) is equivalent to  $h_1 = h_2 = 0$ , where

$$h_1 = \mu z^3 + y^4 z^2 - y^7, \quad h_2 = z^2 + \lambda y^4 z + 3y^3.$$

Thus  $f^{\mu,0,\lambda}$  is singular if and only if there exists  $z \in k^*$  such a that  $h_1 = h_2 = 0$  for some  $y \in k$  satisfying  $y^{10} = 1$ . Let

$$h_3 = (3y^6 h_1 + h_2)/z = 3\mu y^6 z^2 + 4z + \lambda y^4, \quad h_4 = h_3 - 3\mu y^6 h_2 = (4 - 3\mu\lambda)z - 9\mu y^9 + \lambda y^4.$$

Clearly  $h_1 = h_2 = 0$  if and only if  $h_2 = h_4 = 0$ . Define  $a, b, \alpha, \beta$  by  $h_2 = z^2 + az + b$  and  $h_4 = \alpha z + \beta$ . There exists a  $z \in k^*$  such that  $h_2 = h_4 = 0$  for some  $y$  satisfying  $y^{10} = 1$  if and only if  $r = (-\beta)^2 + a\alpha(-\beta) + b\alpha^2$  vanishes for some  $y$  satisfying  $y^{10} = 1$ , for  $b = 3y^3 \neq 0$ . Since  $r = 3y^8\{y^5(16 - 18\mu\lambda) + 27\mu^2 - \lambda^2 + \mu\lambda^3\}$ , there exists a  $y$  satisfying  $y^{10} = 1$  and  $r = 0$ , if and only if  $\{(16 - 18\mu\lambda) + 27\mu^2 - \lambda^2 + \mu\lambda^3\}\{-(16 - 18\mu\lambda) + 27\mu^2 - \lambda^2 + \mu\lambda^3\} = 0$ .

**Lemma 2.5.** The quartic form  $f^{\mu,\nu,0}$  is singular if and only if  $256(1 + 3\mu^2\nu^2)^2 - (27\mu^2 + 6\mu\nu + 27\nu^2 - 16\mu^3\nu^3)^2 = 0$ .

*Proof.* If  $\mu\nu = 0$ , we are done by Lemma 2.2. We write  $f$  for  $f^{\mu,\nu,0}$ . Assume  $\mu\nu \neq 0$  and that  $V(f)$  is singular at  $(x, y, z, t)$ . We can easily show that  $xyzt \neq 0$ . We assume  $t = 1$ . Note that

$$xf_x - zf_z = 3(x^3y - z^3) - (y^3z - x), \quad yf_y - f_t = x^3y - z^3 + 3(y^3z - x).$$

Consequently  $x = y^3z$  and  $x^3y = z^3$ , hence  $y^{10} = 1$ . Now that  $y^{10} = 1$  and  $x = y^3z$ , the condition  $f_x = f_y = f_z = f_t = 0$  is equivalent to  $h_1 = h_2 = 0$ , where  $h_1 = 2\mu z^3 + 3y^4 z^2 + y^7$  and  $h_2 = z^3 + 3y^3 z + 2\nu y^2$ . Let  $h_3 = z^2 - 2\mu y^{-1}z + \frac{1}{3}y^{-2}(-4\mu\nu + \delta)$  ( $\delta = y^5$ ) and  $h_4 = (3\mu^2 + \mu\nu + 2\delta)z + \frac{1}{2}(4\mu^2\nu - \mu\delta + 3\nu\delta)$  so that

$$h_1 = \mu h_2 + 3y^4 h_3, \quad h_2 = (z + 2\mu y^{-1})h_3 + \frac{4}{3}h_4.$$

Note that there exists a  $z \in k^*$  such that  $h_1 = h_2 = 0$  for some  $y$  satisfying  $y^{10} = 1$  if and only if there exists a  $z \in k$  such that  $h_3 = h_4 = 0$  for some  $y$  satisfying  $y^{10} = 1$ . Define  $a, b, \alpha, \beta$  by  $h_3 = z^2 + az + b$  and  $h_4 = \alpha z + \beta$ . Then  $f$  is singular if and only if there exists a  $y \in k$  satisfying  $y^{10} = 1$  and  $r = 0$ , where  $r = (-\beta)^1 + a\alpha(-\beta) + b\alpha^2$ . Since  $12y^2r$  is equal to  $16(1 + 3\mu^2\nu^2)\delta + 27\mu^2 + 6\mu\nu + 27\nu^2 - 16\mu^3\nu^3$ , (1) follows when  $\mu\nu \neq 0$ .

Denote the following polynomial in  $u, v$ , and  $w$  by  $R$ ;

$$\{16 - 18(u + v)w + 48u^2v^2 + 20uvw\}^2 - \{27(u^2 + v^2) + 6uv - w^2 - 36(u + v)uvw + (u + v)w^3 - 16u^3v^3 + 8u^2v^2w^2 - uvw^4\}^2.$$

**Theorem 2.6.** Let the polynomial  $R$  be as above. The quartic form  $f^{\mu,\nu,\lambda}$  is singular if and only if  $R(\mu, \nu, \lambda) = 0$ .

*Proof.* Due to Lemma 2.4 and Lemma 2.5 it suffices to prove the theorem under the condition  $\mu\nu\lambda \neq 0$ . We write  $f$  for  $f^{\mu,\nu,\lambda}$ . Assume that  $V(f)$  is singular at  $(x, y, z, t)$ . Since we can show  $xyzt \neq 0$ , we assume  $t = 1$ . The condition  $f_x = f_y = f_z = f_t = 0$  is equivalent to  $g_j = 0$  ( $j \in [1, 4]$ ), where  $g_1 = xf_x - f_t$ ,  $g_2 = yf_y - f_t$ ,  $g_3 = zf_z - f_t$ ,  $g_4 = f_t$ . Since  $x^3y - z^3 = (3g_1 + g_2 - 3g_3)/10$  vanishes, the equality  $g_2 = 0$  implies  $x = y^3z$ , hence  $y^{10} = 1$ . Now that  $y^{10} = 1$  and  $x = y^3z$ , the condition  $g_1 = g_2 = g_3 = g_4 = 0$  is equivalent to  $h_1 = h_2 = 0$ , where

$$h_1 = \mu z^4 + y^4 z^3 - y^7 z - \nu y^6, \quad h_2 = z^3 + \lambda y^4 z^2 + 3y^3 z + 2\nu y^2.$$

Let  $h_3 = 2\mu z^3 + 3y^4 z^2 + \lambda y^8 z + y^7$  so that  $2y^6 h_1 + h_2 = y^6 z h_3$ . Since we assume  $\mu\nu \neq 0$ , there exists a  $z \in k^*$  such that  $h_1 = h_2 = 0$  for some  $h \in k$  satisfying  $y^{10} = 1$  if and only if there exists a  $z \in k$  such that  $h_2 = h_3 = 0$  for some  $y$  satisfying  $y^{10} = 1$ . As is well known [4, p.203], there exists a  $z \in k$  such that  $h_2 = h_3 = 0$  if and only if  $\det S = 0$ , where

$$S = \begin{bmatrix} 1 & \lambda y^4 & 3y^3 & 2\nu y^2 & 0 & 0 \\ 0 & 1 & \lambda y^4 & 3y^3 & 2\nu y^2 & 0 \\ 0 & 0 & 1 & \lambda y^4 & 3y^3 & 2\nu y^2 \\ 2\mu & 3y^4 & \lambda y^8 & y^7 & 0 & 0 \\ 0 & 2\mu & 3y^4 & \lambda y^8 & y^7 & 0 \\ 0 & 0 & 2\mu & 3y^4 & \lambda y^8 & y^7 \end{bmatrix}.$$

Let  $\delta = y^5$ . Expanding  $\det S$  according to the first column, we have  $\det S = s_1 - 2\mu s_2$ , where

$$s_1 = \begin{vmatrix} 1 & \lambda y^4 & 3y^3 & 2\nu y^2 & 0 \\ 0 & 1 & \lambda y^4 & 3y^3 & 2\nu y^2 \\ 3y^4 & \delta \lambda y^4 & \delta y^2 & 0 & 0 \\ 2\mu & 3y^4 & \delta \lambda y^3 & \delta y^2 & 0 \\ 0 & 2\mu & 3y^4 & \delta \lambda y^3 & \delta y^2 \end{vmatrix}, \quad s_2 = \begin{vmatrix} \lambda y^4 & 3y^3 & 2\nu y^2 & 0 & 0 \\ 1 & \lambda y^4 & 3y^3 & 2\nu y^2 & 0 \\ 0 & 1 & \lambda y^4 & 3y^3 & 2\nu y^2 \\ 2\mu & 3y^4 & \lambda y^8 & y^7 & 0 \\ 0 & 2\mu & 3y^4 & \lambda y^8 & y^7 \end{vmatrix}.$$

We must calculate accurately. We compute  $s_1$  as follows.

$$\begin{aligned} & \begin{vmatrix} 1 & \lambda y^4 & 3y^3 & 2\nu y^2 & 0 \\ 0 & 1 & \lambda y^4 & 3y^3 & 2\nu y^2 \\ 0 & -2\lambda y^8 & -8y^7 & -6\nu y^6 & 0 \\ 0 & (3 - 2\mu\lambda)y^4 & (-6\mu + \delta\lambda)y^3 & (-4\mu\nu + \delta)y^2 & 0 \\ 0 & 2\mu & 3y^4 & \lambda y^8 & y^7 \end{vmatrix} \\ &= y^2 \begin{vmatrix} 1 & \lambda y^4 & 3y^3 & 2\nu y^2 & 0 \\ 0 & 1 & \delta \lambda y^{-1} & 3\delta y^{-2} & 2\nu y^{-3} \\ 0 & -2\lambda & -8y^{-1} & -6\nu y^{-2} & 0 \\ 0 & 3 - 2\mu\lambda & (-6\mu + \delta\lambda)y^{-1} & (-4\mu\nu + \delta)y^{-2} & 0 \\ 0 & 2\mu & 3\delta y^{-1} & \lambda y^{-2} & y^{-3} \end{vmatrix} \\ &= 4y[16 + (-3\mu - 18\nu)\lambda + (4\mu^2\nu^2 + 3\mu\nu\lambda^2) + \delta\{(7\mu\nu + 27\nu^2 - \lambda^2) + (-12\mu\nu^2\lambda + \nu\lambda^3)\}]. \end{aligned}$$

Similarly,  $s_2$  takes the form

$$2y[15\lambda + (-44\mu\nu^2 - 17\nu\lambda^2) + \delta\{(-27\mu + \nu) + (36\mu\nu\lambda + 24\nu^2\lambda - \lambda^3) + (16\nu^2\mu^3 - 8\mu\nu^2\lambda^2 + \nu\lambda^4)\}].$$

Thus  $\det S/(4y)$  is equal to

$$\begin{aligned} & \delta\{(27\mu^2 + 6\mu\nu + 27\nu^2 - \lambda^2) - 36(\mu^2\nu + \mu\nu^2)\lambda + (\mu + \nu)\lambda^3 - 16\mu^3\nu^3 + 8\mu^2\nu^2\lambda^2 - \mu\nu\lambda^4\} \\ & + 16 - 18(\mu + \nu)\lambda + 48\mu^2\nu^2 + 20\mu\nu\lambda^2. \end{aligned}$$

Consequently  $f$  with  $\mu\nu\lambda \neq 0$  is singular if and only if  $R(\mu, \nu, \lambda) = 0$ .



### 3 $\mathbf{Z}_7$ -invariant nonsingular quartic surfaces

We shall describe  $\mathbf{Z}_7$ -invariant nonsingular quartic forms in  $k[x, y, z, t]$ . Let  $p$  be the prime 7,  $\varepsilon \in k^*$  of order  $p$ , diagonal matrices  $D_j$  and  $D_{j,\ell}$  as in the section 1. Let

$$A_0 = D_0, A_2 = D_1, A_2 = D_2, A_3 = D_3, A_4 = D_6, A_5 = D_{2,3}, A_6 = D_{2,4}.$$

A subgroup of  $PGL_4(k)$  isomorphic to  $\mathbf{Z}_p$  is conjugate to one of the seven cyclic groups  $\langle (A_i) \rangle$  by Lemma 1.6. Denote by  $f^{[i,j]}$  ( $i, j \in [0, 6]$ ) the quartic form  $f$  in  $k[x, y, z, t]$  such that  $f_{A_i^{-1}} = \varepsilon^j f$ . Calculating indices of singularity-checking quartic monomials for  $A_i$ , we see easily that  $f^{[i,j]}$  is singular except for the case  $[i, j] = [6, 0]$ . Since  $f^{[6,0]}$  takes the form  $ax^4 + by^3t + cz^3y + dt^3z + exyzt$ , we must have  $abcd \neq 0$ , for  $f^{[6,0]}$  is assumed to be nonsingular. We can easily find a nonsingular diagonal matrix  $D = \text{diag}[\alpha, \beta, \gamma, \delta]$  such that  $f = f_{D^{-1}}^{[6,0]} = x^4 + y^3t + z^3y + t^3z + \lambda xyzt$ . It is immediate that  $g = f_{T^{-1}} = x^3y + y^3z + z^3x + t^4 + \lambda xyzt$  for  $T = [e_4, e_3, e_2, e_1]$  and that  $g_{A^{-1}} = g$  for  $A = TA_6T^{-1} = \text{diag}[\varepsilon^4, \varepsilon^2, \varepsilon, 1]$ .

**Proposition 3.1.** *Let  $G$  be the projective automorphism group of a nonsingular quartic form. In the decomposition  $\Pi p^{\nu(p)}$  of  $|G|$  into prime factors it holds that  $\nu(7) \leq 1$ .*

*Proof.* Let  $d = \nu(7)$  and  $p = 7$ . Assume  $G = \text{Paut}(f)$ , where  $f$  is a nonsingular quartic form. It suffices to show that  $d < 2$  under the assumption  $d \geq 1$ . We may assume  $f = g$  for some  $\lambda \in k$ , hence  $(A) \in G$ , where  $A = \text{diag}[\varepsilon^4, \varepsilon^2, \varepsilon, 1]$ . Suppose  $d \geq 2$ . By Sylow theorem [3] there exists a subgroup  $H$  of  $G$  such that  $(A) \in H$  and  $|H| = p^2$ . We can directly show that any  $X \in GL_4(k)$  satisfying  $(A)(X) = (X)(A)$  is diagonal. Besides,  $H$  is abelian. Thus, if  $(B) \in H$ , we may assume  $B = \text{diag}[\alpha, \beta, \gamma, 1]$ , and it follows that  $\alpha^3\beta = \beta^3\gamma = \gamma^3\alpha = 1$ , hence  $B = \text{diag}[\gamma^{-3}, \gamma^9, \gamma, 1]$  with  $\gamma^{28} = 1$ . Consequently  $|H| \leq 28 < p^2$ , a contradiction.

**Proposition 3.2.** *The quartic form  $f^\lambda = x^3y + y^3z + z^3x + t^4 + \lambda xyzt$  is singular if and only if  $\lambda^4 = 4^4$ .*

*Proof.* It is clear that  $f^0$  is nonsingular. We write  $f$  for  $f^\lambda$ . Assume  $\lambda \neq 0$  and that  $V(f)$  is singular at  $(x, y, z, t)$ . Since  $xyzt \neq 0$ , we may assume  $t = 1$ . Obviously the condition  $f_x = f_y = f_z = f_t = 0$  is equivalent to  $g_j = 0$  ( $j \in [1, 4]$ ), where  $g_1 = xf_x - f_t$ ,  $g_2 = yf_y - f_t$ ,  $g_3 = zf_z - f_t$ , and  $g_4 = f_t$ . In addition the condition  $g_1 = g_2 = g_3 = 0$  is equivalent to  $x^3y = y^3z = z^3x = 1$ . Therefore,  $[x, y, z] \in \mathcal{S}$ , where  $\mathcal{S} = \{[\eta^i, \eta^{-3i}, \eta^{9i}] : i \in [0, 27]\}$  with  $\text{ord}(\eta) = 28$ . Thus  $f^\lambda$  with  $\lambda \neq 0$  is singular if and only if  $g_4 = 0$  for some  $[x, y, z] \in \mathcal{S}$ , i.e.  $\lambda\eta^{7i} + 4 = 0$  for some  $i \in [0, 27]$ . Since  $\text{ord}(\eta^7) = 4$ ,  $f^\lambda$  is singular if and only if  $\lambda^4 = 4^4$ .

### 4 $\mathbf{Z}_p$ -invariant nonsingular quartic surfaces

Let  $p$  be a prime not less than 11,  $\varepsilon \in k^*$  be of order  $p$ . As mentioned in the section 1, a subgroup of  $PGL_4(k)$  isomorphic to  $\mathbf{Z}_p$  is conjugate to one of the subgroups  $\langle (D_0) \rangle$ ,  $\langle (D_1) \rangle$ ,  $\langle (D_\ell) \rangle$  ( $1 < \ell < p$ ),  $\langle (D_{j,\ell}) \rangle$  ( $1 < j \neq \ell < p$ ). Singularity-checking quartic monomials in  $k[x, y, z, t]$  have indices for  $D_0, D_1, D_\ell$  and  $D_{j,\ell}$  as follows.

	$x^4$	$y^4$	$z^4$	$t^4$	$x^3y$	$x^3z$	$x^3t$	$y^3x$	$y^3z$	$y^3t$	$z^3x$	$z^3y$	$z^3t$	$t^3x$	$t^3y$	$t^3z$
$D_0$	0	0	4	0	0	1	0	0	1	0	3	3	3	0	0	0
$D_1$	0	0	4	4	0	1	1	0	1	1	3	3	4	3	3	4
$D_\ell$	0	0	4	$4\ell$	0	1	$\ell$	0	1	$\ell$	3	3	$\ell+3$	$3\ell$	$3\ell$	$3\ell+1$
$D_{j,\ell}$	0	4	$4j$	$4\ell$	1	$j$	$\ell$	3	$j+3$	$\ell+3$	$3j$	$3j+1$	$3j+\ell$	$3\ell$	$3\ell+1$	$3\ell+j$

Recall that  $e_i$  (resp.  $\varepsilon_i$ ) ( $i \in [1, 4]$ ) stands for the  $i$ -th column (resp. row) of the unit matrix  $E_4 \in GL_4(k)$ .

**Lemma 4.1.** (1) Any  $D_0$ -invariant quartic form is singular.  
(2) Any  $D_1$ -invariant quartic form is singular.

*Proof.* Let  $A = D_0$ , and  $f \in k[x, y, z, t]$  a non-zero  $A$ -invariant quartic form. Then  $f_{A^{-1}} = \varepsilon^i f$  for some  $i \in [0, p-1]$ , and  $f$  is a linear combination of quartic monomials of index  $i$  for  $A$ . If  $i = 0$ , then  $f$  contains none of monomials  $z^4, z^3x, z^3y, z^3t$ , hence the projective algebraic set  $V(f)$  is singular at  $(\varepsilon_3)$ . Similarly, according as  $i = 1, i = 3$  or  $i = 4$ ,  $V(f)$  is singular at  $(\varepsilon_3), (\varepsilon_1)$  or  $(\varepsilon_1)$ . If  $i \in [0, p-1] \setminus \{0, 1, 3, 4\}$ , then  $V(f)$  is singular at  $(\varepsilon_1)$ . We can show (2) similarly.

**Lemma 4.2.** Any  $D_\ell$ -invariant quartic form is singular ( $1 < \ell < p$ ).

*Proof.* Let  $A = D_\ell$ , and a non-zero quartic form  $f \in k[x, y, z, t]$  satisfy  $f_{A^{-1}} = \varepsilon^i f$  for some  $i \in [0, p-1]$ . Note that

$$\begin{aligned} \ell &\not\equiv \ell + 3, 3\ell, 4\ell \\ \ell + 3 &\not\equiv \ell, 3\ell + 1, 4\ell \\ 3\ell &\not\equiv \ell, 3\ell + 1, 4\ell \\ 3\ell + 1 &\not\equiv \ell + 3, 3\ell, 4\ell \\ 4\ell &\not\equiv \ell, \ell + 3, 3\ell, 3\ell + 1 \end{aligned}$$

in  $\mathbf{Z}/p\mathbf{Z}$ . Let  $\mathcal{L} = \{\ell, \ell + 3, 3\ell, 3\ell + 1, 4\ell\}$ . Assume first that  $i = 0$ . If  $0 \notin \mathcal{L}$ , then  $V(f)$  is singular at  $(\varepsilon_4)$ . If  $0 \in \mathcal{L}$ , then 0-1)  $\ell + 3 \equiv 0 \not\equiv 3\ell + 1$  or 0-2)  $\ell + 3 \equiv 0 \equiv 3\ell + 1$ , hence  $V(f)$  is singular at  $(\varepsilon_4)$  or  $(\varepsilon_3)$ , respectively. Assume  $i = 3$ . If  $3 \notin \mathcal{L}$ , then  $V(f)$  is singular at  $(\varepsilon_1)$ . If  $3 \in \mathcal{L}$ , then 3-1)  $\ell \equiv 3 \not\equiv 3\ell, 3\ell + 1, 4\ell$  or 3-2)  $\ell \equiv 3$  so that  $V(f)$  is singular at  $(\varepsilon_4)$  or  $\varepsilon_1$ . Assume  $i = 4$ . If  $4 \notin \mathcal{L}$ , then  $V(f)$  is singular at  $(\varepsilon_1)$ . If  $4 \in \mathcal{L}$ , then either 4-1)  $4 \equiv \ell \not\equiv 3\ell$  or 4-2)  $\ell \equiv 3\ell \equiv 4$ , so that  $V(f)$  is singular at  $(\varepsilon_4)$  or  $(\varepsilon_1)$  accordingly. Now assume  $i \in [0, p-1] \setminus \{0, 1, 3, 4\}$ . If  $i \notin \mathcal{L}$ , then  $V(f)$  is singular at any  $(\varepsilon_j)$ , provided a non-zero quartic form  $f$  satisfying  $f_{A^{-1}} = \varepsilon^i f$  exists. Suppose  $i \in \mathcal{L}$ . If  $i \equiv \ell$ , then  $V(f)$  is singular at  $(\varepsilon_3)$ , for  $\ell \equiv i \not\equiv \ell + 3$ . If  $i \not\equiv \ell$ , then  $V(f)$  is singular at  $(\varepsilon_1)$ .

**Lemma 4.3.** Any  $D_{j,\ell}$ -invariant quartic form ( $1 < j \neq \ell < p$ ) is singular.

*Proof.* Let  $A = D_{j,\ell}$  and let  $f \in k[x, y, z, t]$  be a non-zero quartic form satisfying  $f_{A^{-1}} = \varepsilon^i f$  for some  $i \in [0, p-1]$ . Define  $J, L \in (\mathbf{Z}/p\mathbf{Z})^5$  and  $N \in (\mathbf{Z}/p\mathbf{Z})^2$  as follows.

$$\begin{aligned} J &= J(j, \ell) = [j_1, j_2, j_3, j_4, j_5] = [j, j + 3, 3j, 3j + 1, 4j], \\ L &= L(j, \ell) = [\ell_1, \ell_2, \ell_3, \ell_4, \ell_5] = [\ell, \ell + 3, 3\ell, 3\ell + 1, 4\ell], \\ N &= N(j, \ell) = [n_1, n_2] = [3j + \ell, j + 3\ell]. \end{aligned}$$

By Corollary 1.3 it suffices to show that  $f$  contains at most three singularity-checking quartic monomials. To this end we will show that any  $i \in [0, p-1]$  appears at most three times in the 12 components of  $J, L$  and  $N$  and that if  $i$  appears three times there, then  $i \notin \{0, 1, 3, 4\}$ . Note that  $J, L$  and  $N$  possess certain symmetry:  $J(\ell, j) = L(j, \ell)$ ,  $L(\ell, j) = J(j, \ell)$ , and  $N(\ell, j) = [n_2, n_1]$  if  $N(j, \ell) = [n_1, n_2]$ . Besides,  $n_1 \neq n_2$ . By the remark at the beginning of the proof of Lemma 4.2  $i$  appears at most twice in  $J$ , namely 1-1)  $j \equiv i \equiv 3j + 1$ , hence  $j \equiv -1/2 \equiv i$ , or 1-2)  $j + 3 \equiv i \equiv 3j$ , hence  $j \equiv 3/2, i \equiv 9/2$ . Similarly  $i$  appears at most twice in  $L$ , namely 1-3)  $\ell \equiv i \equiv 3\ell + 1$  or 1-4)  $\ell + 3 \equiv 3\ell$ . First assume 1-1) so that  $i \not\equiv \ell, 3\ell + 1$  and 1-4) is impossible. Therefore  $i$  appears at most once in  $L$ , namely 1-1-1)  $i \equiv \ell + 3$ , 1-1-2)  $i \equiv 3\ell$  or 1-1-3)  $i \equiv 4\ell$ . In each of the three cases we can easily see  $i \not\equiv n_1, n_2$ , for  $j, j-1, \ell, \ell-1, j-\ell \not\equiv 0$ . We note that  $-1/2, 9/2 \notin \{0, 1, 3, 4\}$



as elements of the finite field  $\mathbf{Z}/p\mathbf{Z}$ . Similarly, in the case 1-2)  $i$  appears at most once in  $L$ , namely 1-2-1)  $i \equiv \ell$ , 1-2-2)  $i \equiv 3\ell + 1$ , or 1-2-3)  $i \equiv 4\ell$ . In addition in each of the three cases  $i \not\equiv n_1, n_2$ . Next assume  $i$  appears once in  $J$ , namely 2-1)  $j \equiv i$ , 2-2)  $j + 3 \equiv i$ , 2-3)  $3j \equiv i$ , 2-4)  $3j + 1 \equiv i$  or 2-5)  $4j \equiv i$ . If  $i$  does not appear in  $L$ , then  $i$  appears at most twice in  $J$ ,  $L$  and  $N$ . Suppose  $i$  appears once in  $L$ , namely 2- $m$ - $n$ )  $j_m \equiv i \equiv \ell_n$ . Obviously cases 2- $n$ - $n$ ) ( $n \in [1, 5]$ ) are impossible, for  $j \not\equiv \ell$ . We will show that  $i$  appears at most three times in  $J$ ,  $L$  and  $N$  and that if  $i$  appears there three times then  $i \notin \{0, 1, 3, 4\}$ . In view of the symmetry of  $J$ ,  $L$  and  $N$  it suffices to check the 10 cases 2- $m$ - $n$ )  $m < n$ . We can show easily  $i \not\equiv n_1, n_2$  in the cases 2-1-2), 2-1-3), 2-1-5). The case 2-1-4) It is easy to see  $i \not\equiv n_2$ . If  $i \equiv n_1$ , then  $10\ell + 3 \equiv 3\ell + 1$  so that  $i \equiv 13/7 \notin \{0, 1, 3, 4\}$ . The case 2-2-3) It is easy to see  $i \not\equiv n_2$ . If  $i \equiv n_1$ , then  $\ell \equiv 9/7$  and  $i = 27/7 \notin \{0, 1, 3, 4\}$ . We can show easily that  $i \not\equiv n_1, n_2$  in the cases 2-2-4) and 2-2-5). In the remaining cases 2-3-4), 2-3-5), 2-4-5) we can show  $i \not\equiv n_1, n_2$ . By what we have proved and the symmetry of  $J$ ,  $L$  and  $N$  it has been proved that  $i \in [0, p-1]$  appears at most three times in  $J$ ,  $L$  and  $N$  and that if  $i$  appears three times there, then  $i \notin \{0, 1, 3, 4\}$  as an element of  $\mathbf{Z}/p\mathbf{Z}$ .

By Lemma 4.1 through Lemma 4.3 we have following propositions.

**Proposition 4.4.** *Let  $p \geq 11$  be a prime. Then any  $\mathbf{Z}_p$ -invariant quartic form in  $k[x, y, z, t]$  is singular.*

**Proposition 4.5.** *Let  $G$  be a projective automorphism group of a nonsingular quartic form in  $k[x, y, z, t]$ , and  $\Pi p^{\nu(p)}$  the decomposition of the order  $|G|$  into prime factors. Then  $\nu(p) = 0$  if  $p \geq 11$ .*

## 5 The projective automorphism group of the quartic surface $V(x^3y + y^3z + z^3t + t^3x)$

As is known, for given integers  $d, n \geq 3$  there exists a nonsingular  $d$ -form  $g$  such that  $|\text{Paut}(f)| \leq |\text{Paut}(g)|$  for any nonsingular  $d$ -form  $f$  in  $k[x_1, \dots, x_n]$  [5]. Burnside conjectured that  $|\text{Paut}(f)| \leq |\text{Paut}(x^4 + y^4 + z^4 + t^4 + 12xyzt)|$  for any nonsingular quartic form  $f(x, y, z, t)$  [1, §272]. Our method in this section may enable us to prove  $|\text{Paut}(f^{\mu, \nu, \lambda})| \leq |\text{Paut}(x^4 + y^4 + z^4 + t^4 + 12xyzt)|$ . The right-hand side is equal to 1920. Let  $f = x^3y + y^3z + z^3t + t^3x$ ,  $B = \text{diag}[1, \beta, \beta^{-2}, \beta^7]$  with  $\text{ord}(\beta) = 20$ ,  $B' = \text{diag}[\beta, \beta^{-2}, \beta^7, 1]$  and  $C = [e_4, e_1, e_2, e_3]$ . Then  $f_{B^{-1}} = \beta f$ ,  $f_{C^{-1}} = f$ , and  $CBC^{-1} = \beta B^{17}$ . In particular  $G_{80} = \{(B)^i(C)^j : i \in [0, 19], j \in [0, 3]\}$  is a subgroup of  $PGL_4(k)$  of order 80. Let  $G_{20} = \langle (B) \rangle = \langle (B') \rangle$ . We shall show

**Theorem 5.1.**  $\text{Paut}(x^3y + y^3z + z^3t + t^3x) = G_{80}$ .

*Proof.* By definition the projective automorphism group  $\text{Paut}(g)$  of a non-zero form  $g(x_1, \dots, x_n)$  consists of  $(A) \in PGL_n(n)$  such that  $g_{A^{-1}} \sim g$ . Recall that  $\text{Hess}(g) = \det[g_{x_i x_j}]$ . As  $\text{Hess}(g_{A^{-1}}) = (\det A)^2 \text{Hess}(g)_{A^{-1}}$ ,  $g_{A^{-1}} \sim g$  implies  $\text{Hess}(g)_{A^{-1}} \sim \text{Hess}(g)$ , hence  $\text{Paut}(g)$  is a subgroup of  $\text{Paut}(\text{Hess}(g))$ . Let  $h = 3^{-4} \text{Hess}(f)$ , where  $f = x^3y + y^3z + z^3t + t^3x$ . The  $h$  takes the form

$$x^4z^4 + y^4t^4 - 4(x^5zt^2 + x^2y^5t + xy^2z^5 + yz^2t^5) + 14x^2y^2z^2t^2.$$

Hence

$$\begin{aligned} h_x &= 4x^3z^4 - 4(5x^4zt^2 + 2xy^5t + y^2z^5) + 28xy^2z^2t^2, \\ h_y &= 4y^3t^4 - 4(5x^2y^4t + 2xyz^5 + z^2t^5) + 28x^2yz^2t^2, \\ h_z &= 4x^4z^3 - 4(x^5t^2 + 5xy^2z^4 + 2yzt^5) + 28x^2y^2z^2t^2, \\ h_t &= 4y^4t^3 - 4(2x^5zt + x^2y^5 + 5yz^2t^4) + 28x^2y^2z^2t. \end{aligned}$$

Denote by  $\mathcal{S}$  the set of all singular points of the projective algebraic set  $V(h)$  in  $\mathbf{P}^3(k)$ . Clearly  $\mathcal{S}_0 = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$  is a subset of  $\mathcal{S}$ . It is immediate that  $(x, y, z, t) \in \mathcal{S}$  with  $xyzt = 0$  belongs to  $\mathcal{S}_0$ . We shall find all  $(x, y, z, 1) \in \mathcal{S} \setminus \mathcal{S}_0$ . Suppose  $(x, y, z, 1)$  with  $xyz \neq 0$  belongs to  $\mathcal{S}$ , namely  $f_x = f_y = f_z = f_t = 0$ . This condition is equivalent to  $g_j = 0$  ( $j \in [1, 4]$ ), where

$$g_1 = (xf_x - f_t)/4, \quad g_2 = (yf_y - f_t)/4, \quad g_3 = (zf_z - f_t)/4, \quad g_4 = f_t.$$

Since  $10y(-x^2y^4 + z^2) = (g_1 - g_3 + 2g_2)$ , we have  $z^2 = x^2y^4$ . Now  $g_2 = 0$  yields  $y^{10} = 1$ . Put  $z = \sigma xy^2$ , where  $\sigma^2 = y^{10} = 1$ . Then the condition  $g_j = 0$  ( $j \in [1, 4]$ ) is equivalent to  $\ell_1 = \ell_2 = 0$ , where

$$\ell_1 = x^8y^8 - 4x^6y^2\sigma + 4x^2y^5 - y^4, \quad \ell_2 = -2x^6y^2\sigma + 7x^4y^6 - 6x^2y^5 + y^4.$$

Thus  $(x, y, z, 1) \in \mathbf{P}^3(k)$  with  $xyz \neq 0$  belongs to  $\mathcal{S}$  if and only if  $[x, y, z] = [x, y, \sigma xy^2]$  for some  $[x, \sigma, y]$  satisfying  $\sigma^2 = y^{10} = 1$ ,  $\ell_1 = \ell_2 = 0$ . Define  $\ell_3$  and  $\ell_4$  as follows.

$$\begin{aligned} \ell_3 &= (\ell_1 + \ell_2)x^{-2}y^{-8} = x^6 - 6x^4y^4\sigma + 7x^2y^8 - 2y^7 = x^6 - 6\eta y^{-1}x^4 + 7y^2x^2 - 2y^{-3}, \\ \ell_4 &= (\ell_2y^{-2}\sigma + 2\ell_3)y^{-4}\sigma = -5x^4 + x^2(-6 + 14\eta)y^{-1} + (1 - 4\eta)y^{-2}, \end{aligned}$$

where  $\eta = y^5\sigma$ . Evidently  $\ell_1 = \ell_2 = 0$  if and only if  $\ell_3 = \ell_4 = 0$ . Moreover,

$$\ell_3 = -\frac{1}{5}\ell_4(x^2 - \frac{6+16\eta}{5}y^{-1}) + \frac{8(1+\eta)}{25}(-x^2 + y^{-1}).$$

We have found all  $(x, y, z, 1) \in \mathcal{S}$  with  $xyz \neq 0$ . Namely,  $(x, y, z, 1) = (x, y, \sigma xy^2)$  with  $\sigma y^5 = 1$  belongs to  $\mathcal{S}$  if and only if  $x^2 = y^{-1}$ , while  $(x, y, z, 1) = (x, y, \sigma xy^2)$  with  $\sigma y^5 = -1$  belongs to  $\mathcal{S}$  if and only if  $\ell_4/(-5) = 0$ , i.e.  $x^4 + 4y^{-1}x^2 - y^{-2} = 0$ . In the first case there are 20  $(x, y, z, 1)$ 's. In fact, if  $y = \beta^{2i}$  ( $i \in 2[0, 4]$ ), then  $[x, y, z] = [\beta^{-i}, \beta^{2i}, \beta^{3i}]$  ( $i \in 2[0, 4] \cup 10 + 2[0, 4]$ ), and if  $y = \beta^{2j}$  ( $j \in 1 + 2[0, 4]$ ), then  $[x, y, z] = [\beta^{-j}, \beta^{2j}, \beta^{3j}]$  ( $j \in (1+2[0, 4]) \cup (11+2[0, 4])$ ). That is,  $(x, y, z, 1) = (\beta^{-i}, \beta^{2i}, \beta^{3i}, 1)$  ( $i \in [0, 19]$ ). The set of these 20 points is  $\mathcal{S}_1 = G_{20}(1, 1, 1, 1)$ . In the second case there are 40 points. Indeed, let  $u = \sqrt{-2 + \sqrt{5}}$  and  $v = \sqrt{-2 - \sqrt{5}}$ . Since  $x^2 = (-2 + \sqrt{5})y^{-1}$ , we have, as in the first case  $(x, y, z, 1) = (u\beta^{-i}, \beta^{2i}, -u\beta^{3i}, 1)$  or  $(x, y, z, 1) = (v\beta^{-i}, \beta^{2i}, -v\beta^{3i}, 1)$  ( $i \in [0, 19]$ ). Put  $\mathcal{S}_2 = \{(u\beta^{-i}, \beta^{2i}, -u\beta^{3i}, 1) : i \in [0, 19]\}$  and  $\mathcal{S}_3 = \{(v\beta^{-i}, \beta^{2i}, -v\beta^{3i}, 1) : i \in [0, 19]\}$ . Now  $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3$ . Note that  $\mathcal{S}_2 = G_{20}(u, 1, -u, 1)$  and  $\mathcal{S}_3 = G_{20}(v, 1, -v, 1)$ .

Let  $Q_0 = (0, 0, 0, 1)$ ,  $Q_1 = (1, 1, 1, 1)$ ,  $Q_2 = (u, 1 - u, 1)$  and  $Q_3 = (v, 1, -v, 1)$ . Evidently  $H = \text{Paut}(h)$  acts on  $\mathcal{S}$ . We claim that there exists no  $(A) \in H$  such that  $(A)Q_0 \notin \mathcal{S}_0$  so that  $H$  acts on  $\mathcal{S}_0$ . To prove the claim it suffices to show that the tangent cone  $T_{Q_0}$  of  $V(h)$  at  $Q_0$  is not isomorphic to the one  $T_{Q_\ell}$  at  $Q_\ell$  ( $\ell = 1, 2, 3$ ) [7, p.95]. Since  $h(x, y, z, 1) = -4yz^2 + y^4 + (-4x^5z + 14x^2y^2z^2) - 4x^2y^5 + (x^4z^4 - 4xy^2z^5)$ ,  $T_{Q_0}$  is the affine algebraic set  $V(yz^2)$  in  $\mathbf{A}^3(k)$ . Let  $h(x+t, y+t, z+t, t)$  takes the form  $b(x, y, z)t^6 + \text{lower terms of } t$ , where

$$b(x, y, z) = 8(-3x^2 - 3y^2 - 3z^2 + xy + yz + 4xz).$$

Since the symmetric matrix  $[b_{ij}]$  associated with the quadratic form  $b$  is nonsingular,  $b$  is an irreducible polynomial so that  $T_{Q_1} = V(b)$  is not isomorphic to  $V(yz^2)$ .

Let  $w$  stands for  $u$  or  $v$ , so that  $w^4 + 4w^2 - 1 = 0$ . Denoting the non-trivial field automorphism of  $\mathbf{Q}(\sqrt{5})$  over  $\mathbf{Q}$  by  $\tau$ , we have  $\tau(u^2) = v^2$ . Then  $h_{T^{-1}}(x, y, z, t) = h(x+wt, y+t, z-wt, t)$  takes the form  $b(x, y, z)t^6 + \text{lower terms of } t$ , where

$$c(x, y, z) = c_{11}x^2 + c_{22}y^2 + c_{33}z^2 + 2c_{12}xy + 2c_{13}zx + 2c_{23}yz,$$

with

$$\begin{aligned}
c_{11} &= c_{33} = 6w^6 + 40w^4 + 14w^2 - 4 = -44w^2 + 12, \\
c_{22} &= 4w^6 + 14w^4 - 40w^2 + 6 = -32w^2 + 4, \\
c_{12} &= 4w^5 + 28w^3 - 20w = 12w^3 - 16w, \\
c_{13} &= -8w^6 - 20w^4 - 28w^2 = 12(-7w^2 + 1), \\
c_{23} &= -20w^5 - 28w^3 + 4w = 52w^3 - 16w.
\end{aligned}$$

Since the determinant of the symmetric matrix  $[c_{ij}]$  is equal to  $320w^8 + 3540w^6 - 1060w^4 - 116w^2 \neq 0$ , the tangent cone  $T_{Q_\ell}$  ( $\ell = 2, 3$ ), i.e. the affine algebraic set  $V(c)$  is not isomorphic to  $T_{Q_0} = V(yz^2)$ .

Since  $G_{80}$  is a subgroup of  $H$ ,  $H$  acts on  $\mathcal{S}_0$  transitively. Let  $H_0 = \{(A) \in H : (A)Q_0 = Q_0\}$ , and assume  $(A) \in H_0$ . Since  $h_{A^{-1}} \sim h$ , we may assume

$$A = \begin{bmatrix} a & b & c & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & r & 0 \\ \alpha & \beta & \gamma & 1 \end{bmatrix}.$$

Now the condition  $h_{A^{-1}} \sim h$  implies  $A = \text{diag}[a, q, r, 1]$ , hence  $(A) \in G_{20}$ . Thus  $|H| = |\mathcal{S}_0| |H_0| = 80$ , and  $H = G_{80}$ . We have shown  $G_{80} \subset \text{Paut}(f) \subset H \subset G_{80}$ .

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